

Kannan Fixed-Point Theorem On Complete Metric Spaces And On Generalized Metric Spaces Depended an Another Function *

S. Moradi

Faculty of Science, Department of Mathematics
Arak University, Arak, Iran

Abstract

We obtain sufficient conditions for existence of unique fixed point of Kannan type mappings on complete metric spaces and on generalized complete metric spaces depended an another function.

Keywords: Fixed point, contractive mapping, sequentially convergent, sub-sequentially convergent.

1 Introduction

The fixed point theorem most be frequently cited in Banach condition mapping principle (see [4] or [6]), which asserts that if (X, d) is a complete metric space and $S : X \longrightarrow X$ is a contractive mapping (S is contractive if there exists $k \in [0, 1)$ such that for all $x, y \in X$, $d(Sx, Sy) \leq kd(x, y)$) then S has a unique fixed point.

In 1968 [5] Kannan established a fixed point theorem for mapping satisfying:

$$d(Sx, Sy) \leq \lambda [d(x, Sx) + d(y, Sy)], \quad (1)$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$.

Kannan's paper [5] was followed by a spate of papers containing a variety of contractive definitions in metric spaces.

Rhoades [7] in 1977 considered 250 type of contractive definitions and analyzed the relationship among them.

*2000 *Mathematics Subject Classification*: Primary 46J10, 46J15, 47H10.

In 2000 Branciari [3] introduced a class of generalized metric spaces by replacing triangular inequality by similar ones which involve four or more points instead of three and improved Banach contraction mapping principle.

Recently Azam and Arshad [1] in 2008 extended the Kannan's theorem for this kind of generalized metric spaces.

In 2009 [2] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh introduced new classes of contractive functions and established the Banach contractive principle.

In the present paper at first we extend the Kannan's theorem [5] and then extend the theorem due to Azam and Arshad [1] for these new classes of functions.

From the main results we need some new definitions.

Definition 1.1. [2] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. T is said subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Definition 1.2. ([1] or [3]) Let X be a nonempty set. Suppose that the mapping $d : X \longrightarrow X$, satisfies:

- (i) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$ [rectangular property].

Then d is called a generalized metric and (X, d) is a generalized metric space.

For more information can see [1] and [3].

2 Main Results

In this section at first we extend the Kannan's theorem [5] and then extend the Azam and Arshad theorem [1].

Theorem 2.1. [Extended Kannan's Theorem] Let (X, d) be a complete metric space and $T, S : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and

$$d(TSx, TSy) \leq \lambda [d(Tx, TSx) + d(Ty, TSy)], \quad (2)$$

for all $x, y \in X$, then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof. Let x_0 be and arbitrary point in X . We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ (equivalently, $x_n = S^n x_0$), $n = 1, 2, \dots$. Using the inequality (2), we

have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq \lambda [d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)], \end{aligned} \quad (3)$$

so,

$$d(Tx_n, Tx_{n+1}) \leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n). \quad (4)$$

By the same argument,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\leq \dots \leq \left(\frac{\lambda}{1-\lambda}\right)^n d(Tx_0, Tx_1). \end{aligned} \quad (5)$$

By (5), for every $m, n \in \mathbb{N}$ such that $m > n$ we have,

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \dots + d(Tx_{n+1}, Tx_n) \\ &\leq \left[\left(\frac{\lambda}{1-\lambda}\right)^{m-1} + \left(\frac{\lambda}{1-\lambda}\right)^{m-2} + \dots + \left(\frac{\lambda}{1-\lambda}\right)^n\right] d(Tx_0, Tx_1) \\ &\leq \left[\left(\frac{\lambda}{1-\lambda}\right)^n + \left(\frac{\lambda}{1-\lambda}\right)^{n+1} + \dots\right] d(Tx_0, Tx_1) \\ &= \left(\frac{\lambda}{1-\lambda}\right)^n \frac{1}{1 - \left(\frac{\lambda}{1-\lambda}\right)} d(Tx_0, Tx_1). \end{aligned} \quad (6)$$

Letting $m, n \rightarrow \infty$ in (6), we have $\{Tx_n\}$ is a Cauchy sequence, and since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n \rightarrow \infty} Tx_n = v. \quad (7)$$

Since T is a subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So there exists $u \in X$ and $\{x_{n(k)}\}_{k=1}^{\infty}$ such that $\lim_{k \rightarrow \infty} x_{n(k)} = u$.

Since T is continuous and $\lim_{k \rightarrow \infty} x_{n(k)} = u$, $\lim_{k \rightarrow \infty} Tx_{n(k)} = Tu$.

By (7) we conclude that $Tu = v$. So

$$\begin{aligned} d(TSu, Tu) &\leq d(TSu, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) + d(TS^{n(k)+1}x_0, Tu) \\ &\leq \lambda [d(Tu, TSu) + d(TS^{n(k)-1}x_0, TS^{n(k)}x_0)] \\ &\quad + \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(TSx_0, Tx_0) + d(Tx_{n(k)+1}, Tu) \\ &= \lambda d(Tu, TSu) + \lambda d(Tx_{n(k)-1}, Tx_{n(k)}) \\ &\quad + \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(Tx_1, Tx_0) + d(Tx_{n(k)+1}, Tu), \end{aligned} \quad (8)$$

hence,

$$\begin{aligned} d(TSu, Tu) &\leq \frac{\lambda}{1-\lambda} d(Tx_{n(k)-1}, Tx_{n(k)}) + \frac{1}{1-\lambda} \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} d(Tx_1, Tx_0) \\ &+ \frac{1}{1-\lambda} d(Tx_{n(k)+1}, Tu) \xrightarrow{k \rightarrow \infty} 0. \end{aligned} \quad (9)$$

Therefore $d(TSu, Tu) = 0$.

Since T is one-to-one $Su = u$. So S has a fixed point.

Since (2) holds and T is one-to-one, S has a unique fixed point.

Now if T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we conclude that $\lim_{n \rightarrow \infty} x_n = u$ and this shows that $\{x_n\}$ converges to the fixed point of S . \square

Remark 2.2. By taking $Tx \equiv x$ in Theorem 2.1, we can conclude the Kannan's theorem[5].

The following example shows that Theorem 2.1 is indeed a proper extension on Kannan's theorem.

Example 2.3. Let $X = \{0\} \cup \{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\}$ endowed with the Euclidean metric. Define $S : X \rightarrow X$ by $S(0) = 0$ and $S(\frac{1}{n}) = \frac{1}{n+1}$ for all $n \geq 4$. Obviously the condition (1) is not true for every $\lambda > 0$. So we can not use the Kannan's theorem [5]. By define $T : X \rightarrow X$ by $T(0) = 0$ and $T(\frac{1}{n}) = \frac{1}{n}$ for all $n \geq 4$ we have, for $m, n \in \mathbb{N}$ ($m > n$),

$$\begin{aligned} |TS(\frac{1}{m}) - TS(\frac{1}{n})| &= \frac{1}{(n+1)^{n+1}} - \frac{1}{(m+1)^{m+1}} \\ &< \frac{1}{(n+1)^{n+1}} \leq \frac{1}{3} \left[\frac{1}{n^n} - \frac{1}{(n+1)^{n+1}} \right] \\ &\leq \frac{1}{3} \left[\frac{1}{n^n} - \frac{1}{(n+1)^{n+1}} + \frac{1}{m^m} - \frac{1}{(m+1)^{m+1}} \right] \\ &= \frac{1}{3} [|T(\frac{1}{n}) - TS\frac{1}{n}| + |T(\frac{1}{m}) - TS\frac{1}{m}|]. \end{aligned} \quad (10)$$

The inequality (10) shows that (2) is true for $\lambda = \frac{1}{3}$. Therefore by Theorem 2.1 S has a unique fixed point.

In the following theorem we extend the Azam and Arshad theorem [1].

Theorem 2.4. Let (X, d) be a complete generalizes metric space and $T, S : X \rightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and

$$d(TSx, TSy) \leq \lambda [d(Tx, TSx) + d(Ty, TSy)], \quad (11)$$

for all $x, y \in X$, then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof.

□

Remark 2.5. By taking $Tx \equiv x$ in Theorem 2.4, we can conclude the Azam and Arshad theorem [1].

The following example shows that Theorem 2.4 is indeed a proper extension on Azam and Arshad theorem.

Example 2.6. [1] Let $X = \{1, 2, 3, 4\}$. Define $d : X \times X \longrightarrow \mathbb{R}$ as follows:

$$d(1, 2) = d(2, 1) = 3,$$

$$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = 1,$$

$$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 4.$$

Obviously (X, d) is a generalized metric space and is not a metric space.

Now define a mapping $S : X \longrightarrow X$ as follows:

$$Sx = \begin{cases} 2 & ; x \neq 1 \\ 4 & ; x = 1 \end{cases}$$

Obviously the inequality (1) is not holds for S for every $\lambda \in [0, \frac{1}{2})$. So we can not use the Azam and Arshad theorem for S .

By define $T : X \longrightarrow X$ by:

$$Tx = \begin{cases} 2 & ; x = 4 \\ 3 & ; x = 2 \\ 4 & ; x = 1 \\ 1 & ; x = 3 \end{cases}$$

we have

$$TSx = \begin{cases} 3 & ; x \neq 1 \\ 2 & ; x = 1 \end{cases}$$

We can show that

$$d(TSx, TSy) \leq \frac{1}{3} [d(Tx, TSx) + d(Ty, TSy)]. \quad (12)$$

Therefore by Theorem 2.4, S has a unique fixed point.

References

- [1] A. Azam and M. Arshad, *Kannan fixed point theorem on generalized metric spaces*, The J. Nonlinear Sci., 1(2008), no. 1, 45-48.
- [2] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh , *Two Fixed-Point Theorem For Special Mapping*, to appear.

- [3] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces* Publ. Math. Debrecen, 57 1-2(2000), 31-37. 1, 1, 1.2
- [4] K. Goebel and W. A. Kirk, *Topiqs in Metric Fixed Point Theory*, Combridge University Press, New York, 1990.
- [5] R. Kannan, *Some results on fixed points*, *Bull. Calcutta Math. Soc.* 60(1968),71-76. 1, 1.
- [6] M. A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, John Wiley and Sons, Inc., 2001.
- [7] B. E. Rhoades, *A Comparison of Various Definitions of Contractive Mappings*, *Trans. Amer. Math. Soc.* 226(1977), 257-290.

Email:
S-Moradi@araku.ac.ir